MORE ON THE DECOMPOSITION OF TREES INTO ISOMORPHIC SUBTREES

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Abstract

Caro and Schonheim [2] gave a necessary condition for a tree T to be the union of pairwise edge-disjoint subtrees, each isomorphic to a given tree G, and showed that this condition is not sufficient in general.

Answering a question raised in [2], we give a simple characterization of those trees G for which this condition is also sufficient for all trees T.

Our notation is similar to that of Alon in [1]. All graphs considered in this paper are finite and undirected. A graph H is said to have a G-decomposition if it is the union of pairwise edge-disjoint subgraphs, each isomorphic to G. We denote this situation by $G|H$.

We denote by $V(G)$ the set of vertices of G and by $E(G)$ the set of edges of G, and put $e(G) = |E(G)|$. If T is a tree, $C(T)$ is the set of all cut points of T. If $u \in C(T)$ and $\{(u,z_i): 1 \le i \le s\}$ is the set of edges incident with u, then T-u is a forest consisting of trees $T_1,\dots,T_s$ where $z_i \in T_i$ for $1 \le i \le s$. The branches of T at u are the trees $T_i \cup (u,z_i)$, $1 \le i \le s$. If v is a vertex of T, $v \ne u$, let $B(u,v,T)$ denote the unique branch of T at u that contains v. We denote by $\mathcal{B}(u,T)$ the set of all branches of T at u, and put $\mathcal{B}_v(u,T) = \mathcal{B}(u,T)\setminus\{B(u,v,T)\}$. We also define $m(u,T) = \max\{e(B): B \in \mathcal{B}(u,T)\}$.

Denote by $d_{t,k}(u,T)$ the number of branches B at u such that $e(B) \equiv t \pmod k$; the (mod k) branching vector of u is the vector $d_k(u,T) = d_{1,k}(u,T), d_{2,k}(u,T),\dots,d_{k-1,k}(u,T))$. If T has k edges we write $d(u,T)$ instead of $d_k(u,T)$. Notice that in this case $d_{i,k}(u,T)$ is just the number of branches at u having exactly i edges.

Let G be a tree with k edges, and let T be a tree. Denote by $G||T$ the following condition: For every $v \in C(T)$, $d_k(v,T)$ is a linear combination with nonnegative integer coefficients of the vectors $d(u,G)$ $(u \in C(G))$.

In this notation Theorem 2 in [2] gives the following necessary condition for a tree T to have a G-decomposition.

THEOREM A. *If G is a tree and $e(G) > 1$, then for every tree T*

$$G|T \to G||T.$$

The converse is not true in general, and some simple counter-examples are given in [2]. In this paper we give a simple characterization of those trees G for which the converse implication holds for all trees T.

Another characterization of these trees G was given by Alon in Theorem 1 of [1] (see condition (ii) below). This characterization is not so simple and, as noted by the referee, it does not yield an easy algorithm to check whether a given tree G satisfies the converse of Theorem A. The characterization given here supplies such an algorithm.

Let G be a tree with k edges, $k > 1$. Consider the following four conditions:

(i) For every tree T, $G||T \to G|T$.

(ii) For every tree T with k edges, $G||T \to G|T$ (i.e., $G||T \to T$ is isomorphic to G).

(iii) For every two distinct vertices $u,v \in C(G)$, the following condition $C = C(u,v)$ holds.

Condition $C(u,v)$: If $e(\mathcal{B}(u,v,G)) = e(\mathcal{B}(v,u,G))$, then there is a bijection $f\colon \mathcal{B}_v(u,G) \to \mathcal{B}_u(v,G)$ such that for every $B \in \mathcal{B}_v(u,G)$ B is isomorphic to $f(B)$ by an isomorphism that carries u onto v. (The existence of such a bijection f is clearly equivalent to the existence of an isomorphism $g\colon \cup\mathcal{B}_v(u,G) \to \cup\mathcal{B}_u(v,G)$ that maps u onto v.)

(iv) For every $u,v \in C(G)$

$$m(u,G) = m(v,G) \to d(u,G) = d(v,G).$$

(The equality $d(u,G) = d(v,G)$ is clearly equivalent to the existence of a size-preserving bijection $f\colon \mathcal{B}(u,G) \to \mathcal{B}(v,G)$, i.e., a bijection f that satisfies $e(B) = e(f(B))$ for all $B \in \mathcal{B}(u,G)$.)

We shall prove the following theorem.

THEOREM 1. *Let G be a tree with k edges, $k > 1$. Then conditions (i) - (iv) are equivalent.*

Clearly, it is very easy to check if a given tree G satisfies (iv). (In fact, we can devise an algorithm that will do the checking in $O(k)$ steps, assuming a suitable standard presentation of G.)

We note that the equivalence of (ii) and (iv) answers Question 2 raised in [2]. The equivalence of (i) and (ii) is just the assertion of Theorem 1 in [1]. Thus, in order to prove Theorem 1, it remains to show that (ii), (iii) and (iv) are equivalent.

We need two simple lemmas.

LEMMA 1. *If G and T are trees, $e(G) = e(T) = k > 1$ and $G||T$, then for every $v \in C(T)$ there is a $u \in C(G)$ such that $d(v,T) = d(u,G)$.*

Proof. By definition there are $u_1,\ldots,u_m \in C(G)$ and nonnegative integers $x_1,\ldots,x_m$ such that

$$d(v,T) = \sum_{i=1}^{m} x_i d(u_i,G). \tag{1}$$

Define $y = (1,2,\ldots,k-1)$. Clearly, the scalar product of y and $d(v,T)$ is the sum of the sizes of all branches in $\mathcal{B}(v,T)$, which is k. Similarly, the scalar product of y and $d(u_i,G)$ is k for all i, $1 \le i \le m$. Thus (1) implies

$$k = \sum_{i=1}^{m} x_i \cdot k$$

and the result follows. □

LEMMA 2. *Let G be a tree and suppose $u,v \in C(G)$, $u \neq v$. Then*

$$m(u,G) = m(v,G) \tag{2}$$

iff

$$e(\mathcal{B}(u,v,G)) = e(\mathcal{B}(v,u,G)). \tag{3}$$

If (2) (and (3)) hold, then $B(u,v,G)$ is the unique branch at u having $m(u,G)$ edges.

Proof. Clearly $B(u,v,G)$ properly contains every branch in $\mathcal{B}_u(v,G)$, and thus for every $B \in \mathcal{B}(v,G) \setminus \{B(v,u,G)\}$

(4) $$e(B(u,v,G)) > e(B).$$

Similarly, for every $C \in \mathcal{B}(u,G) \setminus \{B(u,v,G)\}$

(5) $$e(B(v,u,G)) > e(C).$$

Suppose (2) holds. If (3) is false, we may assume, without loss of generality, that

(6) $$e(B(u,v,G)) > e(B(v,u,G)).$$

Relations (4) and (6) imply

$$m(v,G) < e(B(u,v,G)) \leq m(u,G),$$

contradicting (2). Thus (2) implies (3). Conversely, if (3) holds, then (4) and (5) imply that

$$m(u,G) = e(B(u,v,G)) = e(B(v,u,G)) = m(v,G),$$

and thus (3) implies (2).

If (3) holds, then (5) implies that $B(u,v,G)$ is the unique branch at u having $m(u,G)$ edges. □

Proof of Theorem 1. We have to prove that conditions (ii), (iii) and (iv) are equivalent. We split the proof into four steps.

Step I. (iii) → (iv).

Assume (iii) holds and suppose $u,v \in C(G)$ satisfy $m(u,G) = m(v,G)$. We have to show that $d(u,G) = d(v,G)$. If $u = v$ this is trivial. Otherwise, $e(B(u,v,G)) = e(B(v,u,G))$ by Lemma 2. By (iii), there is a bijection $f\colon \mathcal{B}_v(u,G) \to \mathcal{B}_u(v,G)$ such that B is isomorphic to

$f(B)$ (and certainly $e(B) = e(f(B))$) for all $B \in \mathcal{B}_v(u,G)$. This, and the fact that $e(B(u,v,G)) = e(B(v,u,G))$, imply that $d(u,G) = d(v,G)$, as needed.

Step II. (iv) → (iii).

We suppose that (iv) holds and prove (iii). Assume (iii) is false, and choose among all pairs of cut points of G that violate condition C a pair (u,v) for which $e(B(u,v,G))$ $(= e(B(v,u,G)))$ is maximal. By Lemma 2 $m(u,G) = m(v,G)$ and thus by (iv), $d(u,G) = d(v,G)$. Since $e(B(u,v,G)) = e(B(v,u,G))$, this means that there is a size-preserving bijection $f\colon \mathcal{B}_v(u,G) \to \mathcal{B}_u(v,G)$. However, u and v do not satisfy condition C, and thus there is a branch $B \in \mathcal{B}_v(u,G)$ that is not isomorphic to $f(B)$ by an isomorphism that carries u onto v. Let u_1 be the unique neighbor of u in B and let v_1 be the unique neighbor of v in $f(B)$. Clearly $u_1,v_1 \in C(G)$,

(7) $$e(B(u_1,v_1,G)) = e(G) - e(B) + 1 = e(G) - e(f(B)) + 1$$
$$= e(B(v_1,u_1,G)),$$

and

(8) $$e(B(u_1,v_1,G)) = e(G) - e(B) + 1 > e(G) - e(B) > e(B((u,v,G)).$$

Because of the extremal choice of u and v (see above), inequality (8) implies condition $C(u_1,v_1)$. Because of (7), this means that there is an isomorphism $g\colon U(\mathcal{B}_{v_1}(u_1,G)) \to U(\mathcal{B}_{u_1}(v_1,G))$ that carries u_1 onto v_1. But this g can clearly be extended to an isomorphism from B to $f(B)$ that carries u onto v. This contradicts the fact that there is not such an isomorphism and thus completes the proof of Step II.

Step III. (ii) → (iv)

Assume (iv) is false. We prove that (ii) is false by constructing a tree T with k edges such that $G||T$ but T is not isomorphic to G. Let $u,v \in C(G)$ satisfy $m(u,G) = m(v,G)$ $(=e(B(u,v,G)))$ but $d(u,G) \neq d(v,G)$. Let T be the tree obtained from G by replacing the branches in $B_v(u,G)$ by isomorphic copies of the branches in $B_u(v,G)$. Thus T consists of two edge-disjoint subtrees T_0, T_1 with a common vertex u: $T_0 = B(u,v,G)$, and T_1 is isomorphic to $UB_u(v,G)$ by an isomorphism r that carries u onto v. Note that the replacement of $UB_v(u,G)$ by T_1 does not affect the vectors $d(x,G)$ for $x \in V(T_0)\setminus\{u\}$.

We first show that $G||T$. Suppose $y \in C(T)$. If $y \in V(T_0)\setminus\{u\}$ then $d(y,T) = d(y,G)$, as noted above. If $y \in V(T_1)\setminus\{u\}$ then $d(y,T) = d(r(y),G)$. Clearly $d(u,T) = d(v,G)$. Thus $G||T$.

Next we show that T is not isomorphic to G, by showing that the number of cut points $y \in C(T)$ for which $d(v,T) = d(v,G)$ is greater than the number of cut points $x \in C(G)$ that satisfy $d(x,G) = d(v,G)$. Indeed, if $x \in V(G)\setminus V(B(u,v,G))$ then clearly $m(x,G) > e(B(u,v,G)) = m(u,G) = m(v,G)$ and thus $d(x,G) \neq d(v,G)$. In addition $d(u,G) \neq d(v,G)$ and thus all the cut points $x \in C(G)$, for which $d(x,G) = d(v,G)$ belong to $V(B(u,v,G))\setminus\{u\}$. However, for each such x, $d(x,T) = d(x,G) = d(v,G)$ and in addition $d(u,T) = d(v,G)$. This completes Step III.

Step IV. (iv) $\rightarrow$ (ii)

Suppose (iv) holds. Let T be a tree with k edges, and suppose $G||T$. We have to show that T is isomorphic to G. Assume this is false. By Lemma 1, for every $v \in C(T)$ there is a $u \in C(G)$ and a size-preserving bijection $f\colon B(u,G) \rightarrow B(v,T)$. Since G is not isomorphic to T, there is a branch $B \in B(u,G)$ such that B is not "properly" isomorphic to $f(B)$, i.e., such that there is no isomorphism $g\colon B \rightarrow f(B)$ that carries u onto v. Let $n(v,u,f)$ denote the smallest possible number of edges of a branch $B \in B(u,G)$ that is not properly isomorphic to $f(B)$. Define $n = \{\min n(v,u,f)\}$, where the minimum is taken over all triples (v,u,f) with $v \in C(T)$, $u \in C(G)$, $d(v,T) = d(u,G)$ and $f\colon B(u,G) \rightarrow B(v,T)$ a size-preserving bijection. Suppose $n = n(v_0,u_0,f_0)$ and let $B \in B(u_0,G)$ be a branch of size n at u_0 that is not properly isomorphic to $C = f_0(B)$. Let u_1 be the unique neighbor of u_0 in B and let v_1 be the

unique neighbour of v_0 in C. Clearly $v_1 \in C(T)$, since otherwise $e(B) = e(C) = 1$, and B would be properly isomorphic to C. Thus, there is a $z \in C(G)$ such that $d(z,G) = d(v_1,T)$, i.e., there is a size-preserving bijection $g\colon B(z,G) \rightarrow B(v_1,T)$.

If $m(v_1,T) < n$, then, by the minimality of n, every $D \in B(z,G)$ is properly isomorphic to $g(D)$, and thus G is isomorphic to T, which contradicts our assumption. Thus $m(v_1,T) \geq n$. Since all the branches of T at v_1 except $B(v_1,v_0,T)$ are properly contained in C and $e(C) = n$, we conclude that $m(v_1,T) = e(B(v_1,v_0,T))$. Clearly

$$n \leq e(B(v_1,v_0,T)) = k - e(C) + 1 = k - e(B) + 1 = e(B(u_1,u_0,G)), \tag{9}$$

and since all the branches of G at u_1 except $B(u_1,u_0,G)$ are properly contained in B, and $e(B) = n$, we conclude that

$$m(u_1,G) = e(B(u_1,u_0,G)) = m(v_1,T).$$

Since $d(z,G) = d(v_1,T)$, clearly $m(z,G) = m(v_1,T) = m(u_1,G)$. Thus, since G satisfies (iv), $d(u_1,G) = d(z,G) = d(v_1,T)$ and therefore there is a size-preserving bijection $h\colon B(u_1,G) \rightarrow B(v_1,T)$. Because of (9) we may assume that

$$h(B(u_1,u_0,G)) = B(v_1,v_0,T).$$

If $D \in B(u_1,G)\setminus\{B(u_1,u_0,G)\}$ then $e(D) < n$ and thus, by the minimality of n, D is properly isomorphic to $h(D)$. Therefore $UB_{u_0}(u_1,G)$ is isomorphic to $UB_{v_0}(v_1,T)$ by an isomorphism that carries u_1 onto v_1. This isomorphism can clearly be extended to a proper isomorphism from B to C, which is impossible, since B and C, as chosen above, are not properly isomorphic. This completes Step IV and establishes Theorem 1. □

Remark. The equivalence between conditions (i) and (iv) of Theorem 1 implies Theorem 2 of [1] and Theorems 3,4,5 and 6 of [2] as special cases.

References

[1] N. Alon, *A note on the decomposition of trees into isomorphic subtrees.* Ars Combinatoria 12 (1981), 117-121.

[2] Y. Caro and J. Schonheim, *Decomposition of trees into isomorphic subtrees.* Ars Combinatoria 9 (1980), 119-130.

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